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STRATIFIED FLOWS AND DIPOLE APPROXIMATIONS*

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The potential method is used to study internal waves behind an obstacle in a three-dimensional ideal incompressible exponentially stratified flow between two horizontal walls. A fundamental solution is constructed for the equations of motion in Boussinesq form, which satisfies the radiation conditions and the boundary conditions on the plane walls. The properties of the fundamental solution and its asymptotic behaviour with respect to the small parameter $\beta = NH/(\pi c)$ are investigated. The fundamental solution is used to construct a solution of the boundary-value problem in the form of a double-layer potential. It is shown that the integral equation for the density is solvable of the parameter β is sufficiently small. Outside the semicylinder containing the perturbation source and aligned parallel to the flow axis, the wave field may be replaced, up to terms of higher order of smallness in the small parameters, by the wave field of a dipole whose moment is a linear function of the sum of the obstacle mass plus the adjoint mass associated with the translational motion of the obstacle along the axis of symmetry parallel to the flow velocity.

1. *Statement of the problem.* An obstacle between two horizontal walls occupies the region Ω. An ideal incompressible stratified fluid flows past the obstacle. The flow depth H/π and the unperturbed flow velocity c are taken as the units of length and velocity. The Vaisala-Brunt frequency N is assumed to be constant. The origin of the coordinate system is located on the top boundary plane, the x axis points in the direction of the flow, and the z axis points vertically upwards. If h is the characteristic size of the region Ω and the parameter $\beta' = Nh/c$ is sufficiently small, then a unique trajectory exists with a bifurcation on the boundary of the region Ω /1/. All other tractories do not have critical points and as $x \to -\infty$ tend asymptotically to straight lines parallel to the x-axis. If $P(x, y, z)$ is an arbitrary point inside the fluid, then its asymptote as $x \to -\infty$ is a distance $\zeta(x, y, z)$ from the xy plane. The problem is to find the function $w(x, y, z) = z - \zeta(x, y, z)$ which specifies the vertical deviation of the trajectory from its asymptote.

The linearized equation for the function w has the form /1/

$$Lw = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial z^2} + \Delta_2\right) w + \beta^2 \Delta_2 w = 0; \quad \beta = \frac{NH}{\pi c}, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.1}$$

The value ζ_0 of the variable $\zeta(x, y, z)$ on the bifurcating trajectory is an unknown constant, which should be determined when solving the problem. The boundary conditions have the form

$$w|_{\partial\Omega} = z|_{\partial\Omega} - \zeta_0 \tag{1.2}$$

$$w|_{z=0} = w|_{z=\pi} = 0, \quad \lim_{r\to\infty} w = \lim_{r\to\infty} |\nabla w| = 0 \quad (r^2 = x^2 + y^2)$$

$$w = o(e^{\gamma x}), \quad |\nabla w| = o(e^{\gamma x}) \quad \text{for} \quad x \to -\infty, \quad \gamma > 0 \tag{1.3}$$

2. ***The fundamental solution and its properties.*** To solve the problem of the flow past an obstacle using potentials, we will use the Fourier method to construct the function $G(x, y, z, \zeta, \beta)$ that satisfies the boundary conditions (1.3) and the equation

$$LG = \delta(z - \zeta)\, \partial^2\delta(x, y)/\partial x^2$$

where $\delta(x, y)$ and $\delta(z - \zeta)$ are delta functions /2/. In the Appendix we show that this function can be reduced to the form

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$$G = (2\pi)^{-1} \beta\theta(x) (H(\beta x, y, z-\zeta) - H(\beta x, y, z+\zeta)) + G_0 + G_1 + G_2 \tag{2.1}$$

$$H(x, y, z) = -\sum_{k=0}^{\infty} J_{2k}(x) S_k(y, z), \quad G_m = G_m(x, y, z, \zeta, \beta), \quad m = 0, 1, 2$$

$$S_0(y, z) = y - \ln 2 - \ln(\operatorname{ch} y - \cos z)$$

$$S_k(y, z) = \sum_{m=0}^{k-1} \frac{C_{k-1}^m}{(m+1)!} y^{m+1} \frac{\partial^{m+1} S_0(y, z)}{\partial y^{m+1}}, \quad k \geqslant 1$$

$$G_0 = -\frac{1}{4\pi} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{\sqrt{r^2 + (z + 2n\pi - \zeta)^2}} - \frac{1}{\sqrt{r^2 + (z + 2n\pi + \zeta)^2}} \right) =$$

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} K_0(nr) \sin(nz) \sin(n\zeta)$$

Here $\theta(x)$ is the Heaviside function, which equals 0 for $x < 0$ and 1 for $x \geqslant 0$, $J_n(x)$ is the Bessel function, and $K_0(nx)$ is the modified Bessel function. The function G_1 is continuously differentiable and satisfies the uniform bound

$$|G_1| + |\nabla G_1| \leqslant C\beta^3 (1 + \beta |x|) \tag{2.2}$$

The function G_2 is continuous, has bounded derivatives, and satisfies the uniform bound

$$|G_2| + |\nabla G_2| \leqslant C\beta^2 \exp(-\sqrt{1 - \beta^2} |x|) \tag{2.3}$$

From (2.1) and (2.2) it follows that the function G has a polar source singularity at the point $(0, 0, \zeta)$ and a logarithmic singularity on the x-axis. The series in the third formula in (2.1) converges rapidly for $\beta x = O(1)$.

3. ***Potential flow.*** Consider the problem of the flow past an obstacle as $\beta \to 0$. The value $\beta = 0$ of the parameter corresponds to homogeneous flow past an obstacle. For small β, we can construct solutions which as $\beta \to 0$ uniformly tend to the solution of the problem of potential flow past an obstacle. First, however, we will investigate the limiting solution for $\beta = 0$.

Let $x + \varphi_1(x, y, z)$ be the potential of the flow past the region Ω. Consider the point source functions for the Dirichlet and Neumann problems in the layer $-\pi < z < 0$,

$$R_0(P, Q) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{r_n(P, Q)} - \frac{1}{r_n(P, Q')} \right) = \tag{3.1}$$

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} K_0\left(n \sqrt{(x-\xi)^2 + (y-\eta)^2}\right) \sin(nz) \sin(n\zeta)$$

$$P = P(x, y, z), \quad Q = Q(\xi, \eta, \zeta), \quad Q' = Q'(\xi, \eta, -\zeta)$$

$$r_n^2 = (x-\xi)^2 + (y-\eta)^2 + (z + 2n\pi - \zeta)^2$$

$$R_1(P, Q) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{r_n(P, Q)} + \frac{1}{r_n(P, Q')} - \frac{1}{n\pi} \right) = \tag{3.2}$$

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} K_0\left(n \sqrt{(x-\xi)^2 + (y-\eta)^2}\right) \cos(nz) \cos(n\zeta)$$

By Parseval's equality, the function R_i is square-integrable in each variable of the points P, Q in the plane layer $T(-\pi < z < 0)$.

Assuming that the surface $\partial\Omega$ satisfies the standard Lyapunov conditions of potential theory, we will seek the function $w_0(Q)$ that solves the external Dirichlet problem for Laplace's equation in the form of a double-layer potential /2/

$$w_0(Q) = \iint_{\partial\Omega} \nu_0(P) \frac{\partial}{\partial N_P} R_0(P, Q)\, dS_P$$

This solution satisfies the conditions on the layer boundaries and the asymptotic conditions at infinity. In order to satisfy conditions (1.2) on the surface $\partial\Omega$, we stipulate that the density $\nu_0(P)$ is the solution of the integral equation

$$\nu_0(Q) + \iint_{\partial\Omega} \nu_0(P) \frac{\partial}{\partial N_P} R_0(P, Q)\, dS_P = z|_{\partial\Omega} - \zeta_0 \tag{3.3}$$

Eq.(3.3) is solvable if and only if the right-hand side is orthogonal to the eigenfunction $\mu_0(P)$ of the conjugate integral equations. If the solution of the homogeneous Neumann problem is sought as a simple-layer potential, then $\mu_0(P)$ is the density of this potential. The corresponding solution of the homogeneous Neumann problem equals 1.

The solvability condition is supplied by the equation for the unknown constant

$$\int_{\partial\Omega} z(P)\mu_0(P)\,dS = \zeta_0 \int_{\partial\Omega} \mu_0(P)\,dS \tag{3.4}$$

The constant ζ_0 can be determined from this equation, i.e., the function $\mu_0(P)$ is not orthogonal to 1.

Indeed, if this were not so the solution of the external Dirichlet problem $w_0^1(P)$ which equals 1 on $\partial\Omega$ could be represented as a double-layer potential. From the square integrability of the kernel over the region T it follows that the functions w_0^1 and ∇w_0^1 are square integrable in the region $T\setminus\Omega$, and the surface integrals of these functions over the lateral surfaces of cylinders of radius R with the generator parallel to the z-axis tend to zero as $R\to\infty$.

Noting that

$$\int_{\partial\Omega} w_0^1 \frac{\partial w_0^1}{\partial N}\,dS = \int_{\partial\Omega} \frac{\partial w_0^1}{\partial N}\,dS = 0$$

and applying Green's formula to w_0^1, we obtain

$$0 = -\int_{T-\Omega} |\nabla w_0^1|^2\,dx\,dy\,dz + \int_{\partial\Omega} w_0^1 \frac{\partial w_0^1}{\partial N}\,dS = -\int_{T\setminus\Omega} |\nabla w_0^1|^2\,dx\,dy\,dz$$

Therefore, $\nabla w_0^1 = 0$ and $w_0^1 = \text{const}$ in the region $T\setminus\Omega$, which is impossible, because the function w_0^1 equals 1 on $\partial\Omega$ and vanishes at infinity. Thus, ζ_0 is determined from Eq. (3.4).

Assume that the region Ω is symmetrical about the straight line parallel to the axis x, the volume of Ω is V, and M is the adjoint mass of the region attributable to translational motion along the axis of symmetry. We can show that

$$\int_{\partial\Omega} \nu_0(P)\cos N\zeta\,dS = \frac{1}{2}(V+M) \tag{3.5}$$

To prove this equality, note that the potential $\varphi_1(Q)$ is the solution of the external Neumann problem and vanishes at infinity. It can therefore be represented as a simple-layer potential. Applying Green's formula to the function $\varphi_1(Q)$, we obtain

$$\varphi_1(Q) = \frac{1}{2}\int_{\partial\Omega}\left(\varphi_1(P)\frac{\partial}{\partial N_P}R_1(P,Q) + R_1(P,Q)\cos N\xi\right)dS_P$$

Let us find the asymptotic behaviour of $\varphi_1(Q)$ as $Q\to\infty$. We can show that

$$R_1(P,Q) \sim \frac{1}{\pi}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{r_n(Q,O)} - \frac{1}{2n\pi} + \frac{x\xi + y\eta}{r_n^3(Q,O)}\right)$$

$$\frac{\partial R_1(P,Q)}{\partial N_P} \sim \frac{1}{\pi}(x\cos N\xi + y\cos N\eta)\sum_{n=-\infty}^{+\infty}\frac{1}{r_n^3(Q,O)}$$

Substituting these relationships into Green's formula, we obtain

$$\varphi_1(Q) = \frac{V+M}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{x}{r_n^3(Q,O)} \tag{3.6}$$

We have previously shown that the function $w_0(Q)$ can be represented as a double-layer potential with density $\nu_0(P)$. The asymptotic expression for $w_0(Q)$ as $Q\to\infty$ has the form

$$w_0(Q) = \frac{1}{\pi}\int_{\partial\Omega}\nu_0(P)\cos N\zeta\,dS\left(\sum_{n=-\infty}^{+\infty}\frac{z+2n\pi}{r_n^3(Q,O)}\right) \tag{3.7}$$

Noting that $v_z = \partial\varphi_1/\partial z = \partial w_0/\partial x$ and comparing formulas (3.6) and (3.7), we obtain (3.5).

4. Flow past an obstacle as $\beta \to 0$. Let

$$R(P, Q, \beta) = -2G(x-\xi, y-\eta, z, \zeta, \beta)$$

We have shown in Sect.2 that the function $R(P, Q, \beta)$ has a source singularity at the point $P = Q$ and a logarithmic singularity $q(x)\ln((y-\eta)^2 + (z-\zeta)^2)$ on the ray $y = \eta, z = \zeta$ ($q(x)$ is a continuously differentiable function). Outside the ray, the function $R(P, Q, \beta)$ is regular in the layer T.

The solution of the problem of non-homogeneous flow past an obstacle will be sought in the form

$$w(Q, \beta) = \int_{\partial\Omega} \nu(P, \beta) \frac{\partial}{\partial N_P} R(P, Q, \beta)\, dS_P \tag{4.1}$$

The right-hand side has the same discontinuity on the surface $\partial\Omega$ as the ordinary double-layer potential, and for the unknown function $\nu(P, \beta)$ we obtain the Fredholm equation

$$\nu(Q, \beta) + \int_{\partial\Omega} \nu(P, \beta) \frac{\partial}{\partial N_P} R(P, Q, \beta)\, dS_P = z|_{\partial\Omega} - \bar{\zeta}, \quad Q \in \partial\Omega \tag{4.2}$$

We have seen in Sect.3 that for $\beta = 0$ and $\bar{\zeta} = \zeta_0$ this equation has the solution $\nu = \nu_0(Q)$. Let

$$\nu = \nu_0 + \beta\nu_1, \quad \bar{\zeta} = \zeta_0 + \beta\zeta_1, \quad R(P, Q, \beta) = R_0(P, Q) + \beta R_1(P, Q, \beta) \tag{4.3}$$

The kernel $\partial R(P, Q, \beta)/\partial N$ generates a completely continuous inegral operator in the space of functions that are continuous on $\partial\Omega$. Let

$$R_i u = -\int_{\partial\Omega} u(P) \frac{\partial}{\partial N_P} R_i(P, Q, \beta)\, dS_P, \quad i = 0, 1 \tag{4.4}$$

$$(u, v) = \int_{\partial\Omega} u(P)\, v(P)\, dS_P$$

Substituting (4.3) into (4.2) and using the notation (4.4), we obtain the equation

$$(I - R_0)\,\nu_1 = -\zeta_1 + R_1\nu_0 + \beta R_1\nu_1 \tag{4.5}$$

Select ζ_1 so that the right-hand side of Eq.(4.5) is orthogonal to μ_0 and normalize μ_0 so that $(\mu_0, 1) = 1$. Then Eq.(4.5) takes the form

$$(I - R_0)\,\nu_1 = R_1\nu_0 + \beta R_1\nu_1 - (\mu_0, R_1\nu_0 + \beta R_1\nu_1)$$

Since the operator $I - R_0$ has a bounded inverse on the subspace of continuous functions orthogonal to μ_0, we obtain

$$\nu_1 = (I - R_0)^{-1}(R_1\nu_0 - (\mu_0, R_1\nu_0)) - \beta(I - R_0)^{-1}(R_1\nu_1 - (\mu_0, R_1\nu_1)) \tag{4.6}$$

For sufficiently small β, Eq.(4.6) has a solution, which can be constructed iteratively. Substituting the expression for ν from (4.3) into (4.1), we obtain

$$w(Q, \beta) = \int_{\partial\Omega} \nu_0(P) \frac{\partial}{\partial N_P} R(P, Q, \beta)\, dS_P + \beta \int_{\partial\Omega} \nu_1(P) \frac{\partial}{\partial N_P} R(P, Q, \beta)\, dS_P \tag{4.7}$$

The second term in (4.7) is omitted, because its contribution is of a higher order of smallness. Then

$$w(x, y, z, \beta) = -2\int_{\partial\Omega} \nu_0(\xi, \eta, \zeta) \frac{\partial}{\partial N_P} G(x-\xi, y-\eta, z, \zeta, \beta)\, dS_P \tag{4.8}$$

Let h_0 be the submersion depth of the obstacle, d its diameter, and T_Ω the region obtained from the layer T by deleting all the rays that originate from the boundary points $\partial\Omega$ in the positive direction of the x-axis. From the properties of the function G described in Sect.2 it follows that the function G is analytical in x, y, z in the region T_Ω for any $(\xi, \eta, \zeta) \in \partial\Omega$.

We assume that the diameter d is small. Expanding the function G in powers of $\zeta - h_0$ for $(\xi, \eta, \zeta) \in \partial\Omega$ and ignoring terms of the order of the diameter squared, we obtain

$$G(x,y,z,\zeta,\beta)=G(x,y,z,h_0,\beta)+\frac{\partial G}{\partial\zeta}(x,y,z,h_0,\beta)(\zeta-h_0)+o((\zeta-h_0)) \tag{4.9}$$

Substituting (4.9) into (4.8) and using the equality (3.5), we obtain that in T_Ω, at distances much greater than the depth of the fluid, we have the formula

$$w(x,y,z,\beta)\sim-(V+M)\frac{\partial G}{\partial\zeta}(x,y,z,h_0,\beta) \tag{4.10}$$

Formula (4.10) may be interpreted as the "dipole approximation" of the flow past an obstacle. Formula (4.10) was obtained in /3/ for the plane problem, when the potential of the corresponding homogeneous flow can be defined by a distribution of dipoles on a horizontal or vertical segment.

5. ***Appendix. Investigation of the properties of the function G.*** Expanding the function G in series in a system of sines $\{\sin(nz)\}$ and taking the Fourier transform of each coefficient of this series with respect to the variables x and y, we can show that

$$G=\theta(x)\sum_{n=1}^{\infty}(G_n^1(x,y,\beta)+G_n^2(x,y,\beta))\sin nz\sin n\zeta \tag{5.1}$$

$$G_n^1=\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\sin(\beta xA(\eta))\cos(\beta y\eta)\frac{A(\eta)}{D(\eta)}d\eta \tag{5.2}$$

$$G_n^2=\frac{1}{2\pi^2}\int_{-\infty}^{+\infty}\exp(-\beta xC(\eta))\cos(\beta y\eta)\frac{C(\eta)}{D(\eta)}d\eta \tag{5.3}$$

$$D(\eta)=\sqrt{(\eta^2+\gamma_n^2)^2+4\eta^2},\quad 2A(\eta)^2=D(\eta)-\eta^2-\gamma_n^2$$
$$2C^2(\eta)=D(\eta)+\eta^2+\gamma_n^2,\quad \gamma_n^2=\varepsilon_n^2-1,\quad \varepsilon_n^2=n/\beta>1$$

As $\varepsilon_n\to\infty$, equalities (5.2) and (5.3) may be rewritten in the form /1/

$$G_n^1=\frac{\beta}{\pi n}\int_{-\infty}^{+\infty}\sin\frac{\beta xt}{\sqrt{t^2+1}}\cos nyt\frac{t\,dt}{(t^2+1)^{3/2}}+G_n^3 \tag{5.4}$$

$$G_n^2=-\pi^{-1}K_0(n\sqrt{x^2+y^2})+G_n^4 \tag{5.5}$$

and for the functions G_n^3 and G_n^4 we have the uniform bounds

$$|G_n^3|\leqslant C_1\beta^3n^{-3}(1+\beta|x|),\quad |G_n^4|\leqslant C_2\beta^2n^{-2}\exp(-x\sqrt{n^2-\beta^2}) \tag{5.6}$$

Substituting (5.4) and (5.5) into (5.1), we obtain (2.1), where the function G_0 is defined by (2.4), the functions G_1 and G_2 satisfy the bounds (2.5) and (2.6), and

$$H=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos nz}{n}\int_{-\infty}^{+\infty}\sin\frac{x\beta t}{\sqrt{t^2+1}}\cos nyt\frac{t\,dt}{(t^2+1)^{3/2}} \tag{5.7}$$

Let us transform formula (5.7). Since the function H is even in y, we may take $y\geqslant 0$. If we introduce a function

$$\chi(\tau)=\sum_{n=1}^{\infty}\frac{e^{in\tau}}{n}=-\ln(1-e^{i\tau})$$

which is regular in the upper halfplane, (5.7) may be rewritten in the form

$$H=-\frac{1}{2\pi\beta}\frac{\partial}{\partial x}\int_{-\infty}^{+\infty}\cos\frac{x\beta t}{\sqrt{t^2+1}}\frac{\chi(z+yt)+\chi(-z-yt)}{t^2+1}dt \tag{5.8}$$

Using the well-known formula

$$\cos(x\beta\cos\varphi)=J_0(\beta x)+2\sum_{k=1}^{\infty}(-1)^kJ_{2k}(\beta x)\cos 2k\varphi$$

we substitute into it the expressions

$$\cos\varphi = \frac{t}{\sqrt{t^2+1}}, \quad \sin\varphi = \frac{1}{\sqrt{t^2+1}}, \quad \cos 2k\varphi = \mathrm{Re}\left(\frac{t+i}{t-i}\right)^k$$

Formula (5.8) may be rewritten in the form

$$H = -\sum_{k=0}^{\infty} J_{2k}(\beta x) S_k(y,z), \quad S_0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\chi(z+yt)+\chi(-z-yt)}{t^2+1}\,dt$$

$$S_k(y,z) = \frac{1}{\pi}\mathrm{Re}\int_{-\infty}^{+\infty} \chi(z+yt)\frac{(t+i)^{k-1}}{(t-i)^{k+1}}\,dt, \quad k \geqslant 1$$

Using the theory of residues to evaluate the integrals, we obtain (2.3).

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LONG-WAVE THERMOCAPILLARY CONVECTION IN LAYERS WITH DEFORMABLE INTERFACES*

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Using non-linear equations describing finite-amplitude deformation of the interfaces /1/ of a system of horizontal immiscible liquid layers, long-wave convective flows are studied for nearly critical Marangoni numbers. The distortion of the interfaces is assumed to be weak. Approximate evolution equations are obtained for the deformation of the interfaces. Analytic solutions describing the stationary surface profile for thermocapillary convection are found, and their stability is investigated.

1. Suppose two horizontal solid plates $(z=0, z=a)$ are maintained at constant and different temperatures, (the temperature difference being equal to θ), and that the space between the plates is filled with two immiscible liquid layers. In equilibrium the thickness of the lower (second) layer is equal to Ha, and that of the higher (first) layer is $(1-H)a$; $0<H<1$. The densities of the media, the coefficients of dynamic and kinematic viscosity, and the thermal conductivity and thermal diffusivity are equal to $\rho_m, \eta_m, \nu_m, \varkappa_m$, and χ_m $(m=1$ for the upper layer and $m=2$ for the lower layer). The surface tension σ depends linearly on temperature T: $\sigma = \sigma_0(1-\alpha T)$.

As units of length, time, velocity, pressure and temperature we take a, a^2/ν_1, ν_1/a, $\rho_1\nu_1^2/a^2$ and θ respectively. In dimensionless variables the convection equations and boundary conditions are written in the form

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